Про Спрямоване Переміщення Графохідного Автомату без Компаса на Графі Квадратної Решітки

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On the Directional Movement of a Graph-Walking Automaton without a Compass on Square Grid Graph

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***Анотація*—Розглянуто задачу організації спрямованого переміщення скінченого автомата без компаса на графі квадратної цілочисельної решітки з позначеними вершинами. Знайдено мінімальну кількість класів позначок необхідну і достатню для того, щоб автомат зберігав довільний напрямок пересування на графі. Розроблено алгоритми побудови мінімальної вершинної розмітки для скінчених та нескінчених решіток.**

***Abstract*—This paper deals with the problem of organizing a directional movement of a finite automaton without a compass on vertex-labeled square grid graph. A minimal number of classes of vertex labels is found that is necessary and sufficient for the automaton to maintain movement direction on the graph. Algorithms for constructing minimal vertex-labeling for finite and infinite grids are developed.**

Ключові слова—граф квадратної решітки; графохідний автомат; вершинна розмітка; спрямоване переміщення

Keywords—square grid graph; graph-walking automaton; vertex-labeling; directional movement

I. INTRODUCTION

Automata walking on graphs are a mathematical formalization of autonomous mobile agents with limited memory operating in discrete environments. Under this model arose an intensively developing broad area of studies of the behaviour of automata in labyrinths (labyrinth is an embedded directed graph of special form) [1, 2]. Research in this regard received a wide range of applications, for example, in the problems of image analysis and navigation of mobile robots [3]. The results for automata and labyrinths are based on the important assumption that automata operating in labyrinths can distinguish directions, that is, they have a compass [4, 5].

In this paper we consider the following problem. Initially the automaton located at an arbitrary vertex of the vertex-labeled square grid graph. The automaton looking over neighborhood of the current vertex and may travel to some neighboring vertex selected by its label. The automaton does not distinguish between equally labeled vertices by their coordinates of direction (that means automaton has no compass). It is required to find necessary and sufficient conditions in the form of restrictions on the properties of the automaton and on the labeling of the graph under which the automaton maintain movement direction.

II. BASIC DEFINITIONS

Let N denote the set of all positive integers and Z the set of integers. For any $n \in N$ we set $Z_n = \{0,1,\dots,n-1\}$.

An infinite square grid graph $G(Z^2)$ is the graph whose vertices correspond to the points in the plane with integer coordinates and two vertices are connected by an edge whenever the corresponding points are at distance 1. It is a regular graph where each vertex has degree 4. We consider the vertex coordinates as its name. A square grid graph $G(Z_n \times Z_m)$ is a subgraph of the $G(Z^2)$ induced by all vertices whose x-coordinates being in the range $0,\ldots,n-1$ and y-coordinates being in the range $0,\ldots,m-1$. We will consider the infinite pass graph as a degenerate square grid graph $G(Z \times Z_1)$.

A labeled graph is a simple connected vertex-labeled graph $G=(V,E,M,\mu)$, where V is a set of vertices, E is a set of edges, M is a set of labels, $\mu : V \to M$ is a mapping. The set of vertices adjacent to a vertex $v \in V$ is called the open neighborhood of v. The neighborhood of v in which v itself is included is called closed neighborhood. A sequence of vertices $p = v_1 \ldots v_k$ where $(v_i, v_{i+1}) \in E$ for $i = 1,\ldots,k-1$ will be called the path in the graph G. We call $k \in N$ the length of the path p. The label $\mu(p)$ of this path is the word $w = \mu(v_1)\ldots\mu(v_k)$ in the alphabet of labels M. We say that the word w is defined by the vertex v_1.

A graph-walking automaton on labeled graph G is a sextuple $A=(S,X,Y,s_0,\varphi,\psi)$, where S is a finite set of internal states, $X = \{(a_0,\{a_1,\ldots,a_k\}) \mid a_i \in M, 0 \le i \le k\}$ is a finite input alphabet (a_0 is a current vertex label, $\{a_1,\ldots,a_k\}$ is a set (or multiset) of labels of all vertices on the current vertex neighborhood, k is a degree of current vertex), $Y = M$ is a finite output alphabet ($y = a$ means that the automaton moves from the current vertex to the adjacent vertex with the label a), $s_0 \in S$ is the initial state, $\varphi : S \times X \to S$ is a transition function, $\psi : S \times X \to Y$ is an output function. Given a labeled graph G, the automaton begins its computation in the state s_0, observing the labeling $\overline{X}_0$ of closed neighborhood of vertex v_0. At each step of the computation, with the automaton in a state $s \in S$ observing a labeling x of closed neighborhood of vertex v, the automaton looks up the transition tables φ and ψ for s and $\overline{X}$. If $\varphi(s,\overline{X})$ is defined as s' and $\psi(s,\overline{X})$ is defined as a, the automaton enters the state s' and moves to the vertex labeled by a. The automaton does not have a compass, that is, it does not distinguish directions and relative position of vertices. Therefore, it does not distinguish vertices with the same labels. It is shown in [6] that automaton without additional resources cannot maintain movement direction on the graph all whose vertices are unlabelled or, equivalently, are labeled with the same label.

Let automaton A at time t be placed at the vertex $v(t)$ of graph $G(Z^2)$. The automaton movement is called uniform and directional if there exists period $T \in N$ that $v(t+T)-v(t) = v(t+2T)-v(t+T)$ or any time \$t\$.

III. Vertex labelling sufficient for directional movement

Let us select on graph $G(Z^2)$ two pairs of opposing directions corresponding to coordinate axes on the plane Z^2. Any automaton trajectory on this graph can be represented as a sequence of moves along these directions. The number of different sequences of labels that automaton must remember increases with the number of directions it can moves along. The complexity of automaton increases as a result. An automaton that can move only in four directions defined by coordinate axes we will consider as a simple one. The computing capabilities of a finite automaton are limited by amount of memory. This leads to restrictions on the labeling of the graph by which automaton moves. First, the alphabet of vertex labels must be finite, and secondly, the words in this alphabet defining the trajectories of the automaton must have a periodic structure. The labeling of graph G can be considered as function $\mu : Z^2 \to M$. A labeling μ of $G(Z^2)$ is periodic in direction $(q,t) \in Z^2$ if $\mu(i+q, j+t) = \mu(i,j)$ for all $i,j \in Z$. We call a labeling traversable if an automaton can move on the graph in any direction with its use. A vertex labeling that minimizes the number of different label types is called a minimal labeling.

Labeling function is called deterministic if all vertices in closed neighborhood of every vertex have different labels. We call labeled graph deterministic if its labeling function is deterministic. From the definition of deterministic graph it follows: (1) for any vertex, every word in the alphabet of labels defines at most one path from this vertex; (2) the distance between two equally labeled vertices is at least 4 (see [7] for more details). These properties provide a principled opportunity for targeted movement of the graph-walking automaton on a deterministic graph. For example, it is possible to construct an automaton moving along paths connecting the vertices of a graph if the labels of these paths are known.

Theorem 1. For a minimal traversable deterministic labeling of the path graph $G(Z \times Z_1)$ it is necessary and sufficient to have three types of labels.

Let us build a minimal traversable deterministic labeling of the graph $G(Z \times Z_1)$. Without loss of generality, assume the $M = \{0,1,2\}$. We define the required labeling by the following condition: for any vertex v if $\mu(v) = a$, $a \in M$, then vertices adjacent to v have labels $b = a \oplus_3 1$ and $c = a \oplus_3 (-1)$ where $\oplus_3$ denotes addition modulo 3. The obtained labeling is periodic in the direction 3. An automaton using this labeling can move in two opposite directions, which we will conditionally call “east” and “west”. Suppose the automaton is at the vertex labeled by a. Then in order to move to the “east”, it must moves to the vertex labeled by b, and in order to move to the “west” – to the vertex labeled by c.

Theorem 2. For a minimal traversable deterministic labeling of the square grid graph:

1) $G(Z \times Z_2)$ it is necessary and sufficient to have four types of labels;

2) $G(Z^2)$ it is necessary and sufficient to have five types of labels.

Let us build a minimal traversable deterministic labeling of the graph $G(Z \times Z_2)$. Assume $M = \{0,1,2,3\}$. We define the required labeling by the following condition: for any vertex v if $\mu(v) = a$, $a \in M$, then vertices adjacent to v have labels $b = a \oplus_4 1$, $c = a \oplus_4 (-1)$ and $d = a \oplus_4 2$, where $\oplus_4$ denotes addition modulo 4. As an example, let $(i,0) \in Z^2$ is an arbitrary vertex and $\mu(i,0) = a$, $\mu(i+1,0) = b$, $\mu(i-1,0) = c$, $\mu(i,1) = d$. The obtained labeling is periodic in the direction $(4,0)$. Under this labeling the automaton movement along abscissa axis from vertex $(i,0)$ are similar to the movements eastward and westward on the graph $G(Z \times Z_1)$. The automaton movement along ordinate axis is to move to a vertex labeled by d.

Let us build a minimal traversable deterministic labeling of the graph $G(Z^2)$. Assume $M = \{0,1,2,3,4\}$. We define the required labeling by the following condition: for any vertex v if $\mu(v) = a$, $a \in M$, then vertices adjacent to v have labels $b = a \oplus_5 1$, $c = a \oplus_5 (-1)$, $d = a \oplus_5 2$ and $e = a \oplus_5 (-2)$, where $\oplus_5$ denotes addition modulo 5. As an example, let $(i,j) \in Z^2$ is an arbitrary vertex and $\mu(i,j) = 0$, $\mu(i+1,j) = 1$, $\mu(i-1,j) = 4$, $\mu(i,j+1) = 2$, $\mu(i,j-1) = 3$ (see Figure 1). The obtained labeling is periodic in directions $(5,0)$ and $(0,5)$. An automaton using this labeling can move in two pairs of opposite directions, which we will conditionally call "east", "west", "north" and "south". Suppose the automaton is at the vertex labeled by a. Then in order to move to the "east", it must moves to the vertex labeled by b, to the "west" – c, to the "north" – d, to the "south" – e.

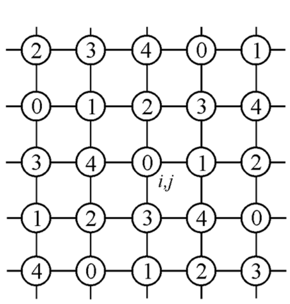

Fig.1. A minimal deterministic labeling of the square grid graph.

Corollary 1. For a minimal traversable deterministic labeling of the square grid graph $G(Z_n \times Z_m)$, $n > 2$, $m > 2$, it is necessary and sufficient to have five types of labels.

We will consider a graph traversal as any path passing through all vertices of the graph. It is shown that there exists an automaton that traverses any graph $G(Z_n \times Z_m)$ with minimal deterministic labeling. A single graph-walking automaton without any enhancements cannot traverse infinite graph. In studies of behavior of automata in labyrinths, one of enhancements consists of permission to drop and lift pebbles at the vertices. In essence, the permission to drop pebbles means that the automaton possesses an unbounded external memory, which greatly increases its possibilities. It is shown that there exists an automaton with two pebbles that traverses any graph $G(Z \times Z_n)$ with minimal deterministic labeling, where $n \geq 1$. It is also shown that there exists an automaton with three pebbles that traverses graph $G(Z^2)$ with minimal deterministic labeling.

IV. Minimal traversable labeling

In this section we discuss problem: is it possible to build a traversable labeling of square grid graph with fewer types of labels than deterministic labeling?

We call a path $p = v_1 \ldots v_k$ deterministic if its labeling $\mu(p) = a_1 \ldots a_k$ satisfies condition: for any path vertex v_i there exists unique adjacent vertex labeled by a_{i+1} where $i = 1, \ldots, k$. It is shown that for the traversability of graph labeling it is necessary and sufficient that for any vertex there exists a deterministic path to all vertices from its neighborhood.

Theorem 3. For a minimal traversable labeling of the path graph $G(Z \times Z_1)$ it is necessary and sufficient to have two types of labels.

Let us build a minimal traversable labeling of the graph $G(Z \times Z_1)$. Assume $M = \{0,1\}$. We define the required labeling by following condition: for any vertex v if $\mu(v) = a$, $a \in M$, then vertices adjacent to v have labels a and $b = a \oplus 1$. The obtained labeling is periodic in direction 4. An automaton using this labeling can move in two opposite directions: first – to the vertex whose label coincides with the label of current vertex and second – to the vertex whose label different from the label of current vertex. We will conditionally call first direction "west" and second direction – "east". Let $i, i+1 \in Z$ are arbitrary vertices and $\mu(i) = \mu(i+1)$. Then "east" for the automaton at the vertex i means "west" for the automaton at the vertex $i+1$ and vice versa. This is the difference between considered labeling and deterministic labeling where at any vertex both directions are uniquely defined. The automaton movement "eastward" consists in the sequential repetition of two steps: (1) move to the vertex whose label different from the label of current vertex; (2) move to vertex whose label coincides with the label of current vertex. Movement "westward" is obtained by interchanging of steps (1) and (2).

Let $G(Z_n \times Z_m)$ be arbitrary finite square grid graph where $n > 2$, $m \geq 2$. It is shown that traversable labeling of $G(Z_n \times Z_m)$ using two types of labels does not exist.

Theorem 4. For a minimal traversable labeling of the square grid graph $G(Z^2)$ it is necessary and sufficient to have three types of labels.

Let us build a minimal traversable labeling of the graph $G(Z^2)$. Assume $M = \{0,1,2\}$. One of possible labeling of this graph defines by following conditions: if vertices (i,j), $(i+1,j)$, $(i-1,j) \in Z^2$ labeled by $a \in M$ then vertices $(i,j+1)$, $(i-1,j+1)$, $(i-1,j-1)$, $(i+2,j)$ labeled by $b = a \oplus_3 1$ and vertices $(i,j-1)$, $(i+1,j+1)$, $(i+1,j-1)$, $(i-2,j)$ labeled by $c = a \oplus_3 (-1)$ (see Figure 2). The obtained labeling is periodic in directions $(9,0)$ and $(0,9)$. It is shown that under this labeling for any vertex there exists a deterministic path to all vertices from its neighborhood. The sequence of such paths determines the movement in the required direction.

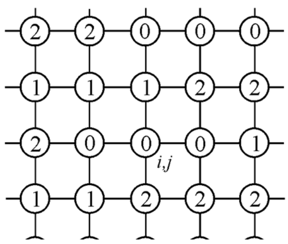

Fig.2. A minimal labeling of the square grid graph.

Corollary 2. For a minimal traversable labeling of the square grid graph $G(Z_n \times Z_m)$, where $n > 2$, $m \geq 2$, it is necessary and sufficient to have three types of labels.

It is shown that there exists an automaton that traverses any graph $G(Z_n \times Z_m)$ with minimal labeling. It is also shown that there exists an automaton with three pebbles that traverses graph $G(Z^2)$ with minimal labeling.

An automaton built to move using minimal traversable labeling is more complicated than an automaton that uses deterministic labeling for movement. The reason is that the first automaton may need to go through several other vertices in order to get from the current vertex to an adjacent one. The automaton trajectories are particularly complicated for vertices that define external face of a finite graph. Hence minimizing traversable labeling of the graph leads to an increase in complexity of the graph-walking automaton.

V. Building Minimal Traversable Labeling

Due to the applicability of both types of minimal labeling to organization of a directional movement of a graph-walking automaton the following problem arises: is it possible to construct an automaton that is capable of constructing a minimal labeling (or deterministic labeling) of an unlabeled graph? Theorems 5 and 6 give positive answers to the question of this problem.

Theorem 5. There exists an automaton that build minimal traversable labeling (deterministic labeling) on path graph $G(Z \times Z_1)$.

Theorem 6. There exists an automaton with three pebbles that build minimal traversable labeling (deterministic labeling) on square grid graph $G(Z^2)$.

Here similar to that of preceding section the automaton building a minimal labeling is more complicated than the automaton building a deterministic labeling.

Conclusion

Necessary and sufficient conditions in the form of restrictions on the properties of the automaton and on the labeling of the graph under which the automaton without a compass maintains movement direction are obtained. Two types of automaton traversable vertex labeling of the graph are proposed. Methods and algorithms of automaton traversal of finite and infinite graphs and building both types of labeling for unlabeled graphs are developed. The obtained results lay the basis for studying navigation of automata without a compass and their collectives in stationary homogeneous discrete environments.

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