Поведінка Безкомпасних Автоматів на Драбинних Графах

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Behavior of Compassless Automata on Ladder Graphs

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***Анотація*—Мета цієї публікації – дослідити можливості колективів безкомпасних автоматів у дослідженні драбинних графів. Кожен граф є анонімним, тобто вершини графа не мають ідентифікуючих позначок, і, таким чином, всі вершини одного степеня здаються автоматам ідентичними. Автомати не розрізняють вершини на основі їх координат або напрямку (це означає, що автомати не мають компаса). Розглядаються колективи, що складаються з керуючого автомата і декількох камінчиків, які є автоматами найпростішої форми, положення яких повністю визначаються керуючим автоматом. Представлено мінімальні колективи, які досліджують драбинні графи та деякі їх підграфи.**

***Abstract*— The objective of this note is to examine the capabilities of collectives of compassless automata in exploring ladder graphs. Each graph is anonymous, meaning the vertices of the graph lack identifying labels, and thus, all vertices of the same degree appear identical to the automata. The automata do not distinguish between vertices based on their coordinates or direction (which means the automata have no compass). We considered collectives consisting of a controlling automaton and some pebbles, which are automata of the simplest form, whose positions are entirely determined by the controlling automaton. We have presented minimal collectives that explore ladder graphs and some of their subgraphs.**

Ключові слова—драбинний граф, колектив автоматів, дослідження графа

Keywords—ladder graph, collective of automata, graph exploration

I. Introduction

Automata walking on graphs are a mathematical formalization of autonomous mobile agents with limited memory operating in discrete environments. Studies investigating the behavior of such automata in finite and infinite labyrinths – embedded directed graphs of a specific form – have emerged and are rapidly evolving within this framework [1, 2, 3]. This research has diverse applications, including image analysis [4, 5] and mobile robotics navigation [6]. The assumption that automata navigating labyrinths can discern directions, i.e., possess a compass, is fundamental to the findings concerning automata and labyrinths [7, 8].

The aim of this note is to investigate the behavior of collectives of compassless automata on ladder graphs. Each automaton receives information about the presence or absence of other automata at the neighboring vertices when operating on the graph. Based on this input, it moves to one of these vertices. The automata do not differentiate between the directions or positions of the vertices, but they can distinguish between occupied and unoccupied vertices. This means each automaton has no compass. This limitation in automata capabilities makes their behavior on graphs more complicated. In a previous publication [9], the author demonstrated that a collective consisting of one controlling automaton and four controllable automata, or pebbles, is a minimal collective capable of maintaining the direction of motion on an infinite square lattice of width 2. Such a lattice is an embedding of an infinite ladder graph on the integer plane.

II. Basic Definitions

Let I be a set of indices and $\{M_i\}_{i\in I}$ be a family of sets. Then, by $T(\{M_i\}_{i\in I})$, we denote the set of all partial transversals of this family, where a partial transversal is defined as containing at most one element from each M_i. Note that the empty set is also a partial transversal.

The path graph P_n is a tree with two vertices of degree 1, and the other $n-2$ vertices of degree 2. The *n*-ladder graph can be defined as $L_n = P_2 \times P_n$ where × denotes the Cartesian product of graphs [10].

A labeled graph is a simple, connected, vertex-labeled graph $G = (V, E, M, \mu)$, where V is a set of vertices, E is the set of edges, M is a finite set of labels, and $\mu : V \to M$ is a mapping.

A graph-walking automaton on labeled graph $G=(V,E,M,\mu)$ is a sextuple $A=(S,X,Y,s_0,\varphi,\psi)$, where S is a finite set of internal states, X is a finite input alphabet consisting of letters x of the form $(\mu(v),\mu(N_v))$ where $\mu(v)\in M$ is the label of a vertex $v\in V$, and $\mu(N_v)$ is a set or multiset of labels of this vertex neighborhood, $Y=M\cup\{\theta\}$ ($\theta\notin M$) is a finite output alphabet where $y=a$ means the automaton moves from the current vertex to an adjacent vertex with label a, and $y=\theta$ means the automaton stays at the current vertex, $s_0\in S$ is an initial state, $\varphi: S\times X\to S$ is a transition function, and $\psi: S\times X\to Y$ is an output function. Given a labeled graph G, the automaton begins its computation in the state s_0, observing the labeling x_0 of the closed neighborhood of vertex v_0. At each step of the computation, when the automaton is in state s and observing the labeling x of the closed neighborhood of vertex v, it consults the transition tables φ and ψ for s and x. If $\varphi(s,x)$ is defined as s' and $\psi(s,x)$ is defined as a, the automaton transitions to state s' and moves to a vertex labeled with a. The automaton does not have a compass, meaning it does not distinguish directions or the relative positions of vertices. Consequently, it does not differentiate between vertices with the same labels.

Let us denote by v_0 the vertex of the graph G at which the automaton A, initially in state s_0, is set. Define $Int(A,G)$ as the set of vertices visited by the automaton. We say that the automaton explores G if $Int(A,G)=V$ for any v_0. G is a trap for A if the automaton fails to explore the graph.

In this note, we focus on the behavior of graph-walking automata on graphs with equally labeled, or equivalently, unlabeled vertices (anonymous / homonymous setting). Clearly, any such graph will serve as a trap for a single compassless automaton, except in the trivial cases. This poses the problem of possibly enriching the automaton model to solve the exploration problem. The most natural approach is a system of interacting automata, referred to as a collective.

Let $I=\{1,2,\dots,n\}$. A system of automata $\mathbf{A}=(A_1,A_2,\dots,A_n)$ where $A_i=(S_i,X_i,Y_i,s_0^i,\varphi_i,\psi_i)$, $1\le i\le n$, is called a collective of automata if the following conditions are satisfied: (1) the input alphabet X_i of the automaton A_i consists of letters x of the form (α,β) where $\beta=\{\beta_1,\dots,\beta_k\}$, $\alpha,\beta_1,\dots,\beta_k\in T(\{S_j \mid j\in I\setminus\{i\}\})$, α describes the automata located on some vertex v, β_1, …, β_k describe the automata located on vertices from the neighborhood of v, and k is the degree of v; (2) $Y_i=D_i\cup\{\theta\}$ where $D_i=T(\{S_j \mid j\in I\setminus\{i\}\})$; (3) $\psi_i(s,a)\in Pr_2(a)\cup\{\theta\}$ for any $s\in S_i$ and $a\in X_i$. Let $J\subset I$. A subsystem $(A_j)_{j\in J}$ of a collective $\mathbf{A}=(A_1,A_2,\dots,A_n)$ is called automata-pebbles (or pebbles) in this collective if for all $j\in J$ the following conditions hold: (1) A_j has a single internal state; (2) A_j can only move if there is an automaton A_i, $i\notin J$, on the same vertex, and A_j can only move to the same vertex as A_i. We assume that the automata can distinguish between all pebbles. If a collective of automata $\mathbf{A}=(A_1,A_2,\dots,A_n)$ has k automata-pebbles, we will refer to it as a collective of type $(n-k,k)$.

The (nondeterministic) behaviour of a collective $\mathbf{A}=(A_1,A_2,\dots,A_n)$ of type $(n-k,k)$ on the graph G is the set $\Pi(A,G)$ of sequences $\pi(A,G)$: $(\vec{x}_0,\vec{s}_0,\vec{y}_0)$, …, $(\vec{x}_t,\vec{s}_t,\vec{y}_t)$, $(\vec{x}_{t+1},\vec{s}_{t+1},\vec{y}_{t+1})$, …, where $\vec{x}_t=(x_t^1,\dots,x_t^n)$, $x_t^i=(\alpha_t^i,\beta_t^i)\in X_i$, $\vec{s}_t=(s_t^1,\dots,s_t^n)$, $s_t^i\in S_i$, $\vec{y}_t=(y_t^1,\dots,y_t^n)$, $y_t^i\in Y_i$, $1\le i\le n$, such that $s_{t+1}^i=\varphi_i(s_t^i,x_t^i)$ and $y_t^i=\psi_i(s_t^i,x_t^i)$. A single sequence $\pi(A,G)$ is referred to as an implementation of the behavior $\Pi(A,G)$.

We further assume that a collective $\mathbf{A}$ explores a graph G if $\mathbf{A}$ explores G for all implementations of its behavior $\Pi(A,G)$. Otherwise, we assume that it does not explore G.

III. Automata Exploration of Ladder Graphs

Below, we describe minimal collectives of automata that explore finite ladder graphs and some of their subgraphs. It is assumed that all automata from the collective $\mathbf{A}$ are initially placed on the same vertex of the graph G.

Theorem 1. The following statements are fulfilled:

(1) A single automaton can explore the graph P_2 but cannot explore the graphs P_3 and L_2.

(2) A collective consisting of one automaton and one pebble can explore the graphs P_3 and L_2 but cannot explore the graphs P_4 and L_3.

(3) A collective consisting of one automaton and two pebbles can explore the graph P_4 but cannot explore the graphs P_5 and L_3.

(4) A collective consisting of one automaton and three pebbles can explore the graph P_n for all n and the graph L_3 but cannot explore the graph L_4.

(5) A collective consisting of one automaton and four pebbles can explore the graph L_n for all n.

In order to address the problem of exploring infinite graphs, it is necessary to introduce additional pebbles to delineate the boundaries of the explored subgraph. In this context, the exploration algorithm entails a collective of automata moving sequentially from one boundary pebble to another, while simultaneously relocating these pebbles in opposite directions.

Theorem 2. The following statements are fulfilled:

(1) A collective of one automaton and five pebbles can explore a 2-way infinite path graph.

(2) A collective of one automaton and six pebbles can explore a 2-way infinite ladder graph.

Conclusion

In this note, we have presented minimal collectives consisting of an automaton and a few pebbles that explore ladder graphs and some of their subgraphs. In future work, we plan to investigate the behavior of collectives of compassless automata on various grid graphs. Such graphs are interesting from the perspective of applying the study's results to the navigation of autonomous mobile robots.

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