Linear Synthesis Problem as a Tool for Optimal Control

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***Abstract* - In today's world, effective management is becoming the key to success. The problem of linear synthesis, as a tool for optimal management, solves the problem of optimizing resources in the face of change. This article explores the process of synthesizing control systems with random factors. It considers variational problems where new functionalities are taken into account to achieve optimal control under uncertainty. The multi-purpose nature of the synthesis process as a key stage in the creation of effective control systems is also investigated.**

Keywords— optimal control system, linear synthesis problem, dynamic systems, mathematical models.

I. Introduction

Linear control synthesis is an important component of control theory that can solve a variety of problems in modern technologies and engineering systems. Its relevance is explained by a wide range of applications in various fields, such as robotics, industrial automation, and power generation. The task of linear synthesis is to create optimal control for systems with dynamic characteristics. The simplest variant of dynamic systems to analyze are those with linear characteristics, which have limitations in the implementation of complex control functions and adaptation to changing conditions. This can lead to instability and inefficiency of control. However, the use of linear synthesis allows us to solve these problems by developing optimal control signals that minimize the impact of constraints and ensure stable and efficient operation of the system. This is achieved through the development of mathematical models and the definition of control signals aimed at minimizing certain quality criteria. The use of linear synthesis increases the efficiency of management, ensuring the stability and accuracy of the system's response to external influences. Such studies are of great importance for both scientific research and practical applications, contributing to the development of new technologies and improving the efficiency of technical systems.

II. The Problem of Synthesizing an Optimal Control System

The task of building a control system is usually solved with a large number of hypotheses. One of the most common is the linearization hypothesis.

$$\dot{x} = A \cdot x + F(t) + \varphi, \tag{1}$$

where $A = \|a_{ij}\|$ is a matrix of specified time functions that depend on changes in the system state; $F(t) = \{F_i(t)\}$ – is a random vector function of the process on time, for which the mathematical expectation $M(F_i(t)) = 0$; φ – is a vector used to change the state of the system x, where $x(0) = x_0$.

Since the right-hand side of Equation (1) includes some stochastic function, the vector x also becomes a stochastic function. As a consequence, any deterministic characteristic *F(x)* of the phase trajectory is also a stochastic quantity. Therefore, it makes sense to consider the mathematical expectation of these quantities as functionalities characterizing control $M(F_i(t))$.

Mathematical expectation of the correction vector:

$$M(\varphi) = M(x, Rx)_{t=T}, \tag{2}$$

where $R = \|r_{ij}\|$ is a given symmetric matrix.

The optimal control in the class of controls that are linear functions of phase coordinates can be represented as equation:

$$W = C \cdot x, \tag{3}$$

where $C = \|c_{ij}(t)\|$ is the feedback coefficients matrix.

Accordingly, equation (1) can be represented as follows:

$$\dot{x} = A(t) \cdot x + C(t) \cdot x + F(t) = D \cdot x + F(t), \tag{4}$$

where $D = \|d_{ij}(t)\|, D = A + C$, $d_{ij}^{\min}(t) \le d_{ij}(t) \le d_{ij}^{\max}(t)$, and $d_{ij}^{\min}(t) = a_{ij}(t) + c_{ij}^{\min}(t)$, $d_{ij}^{\max}(t) = a_{ij}(t) + c_{ij}^{\max}(t)$.

Accordingly, the problem of optimal management:

$$\dot{p} = -\tilde{D} \cdot p, \tag{5}$$

where $\tilde{D}$ is the Hermitian conjugate matrix to the matrix D.

Using (6), we find $\frac{d}{dt}(p, x) = (p, F)$:

$$(p(T), x(T)) = (p(0), x(0)) + \int_0^T (p(T), F(T))dt. \tag{6}$$

Consider a vector that satisfies equation (5) and the conditions:

$$p_i^s(T)=\begin{cases}1, \text{if } t=T\\ 0, \text{if } t\neq T\end{cases}, \tag{7}$$

accordingly,

$$x_i(T)=\left(p_i(0),x(0)\right)+\int_0^T\left(p_i(t),F(t)\right)dt, i=1,...n.$$

Equation (2) will have the form:

$$M(\varphi)=\sum_{\substack{i=1,\\ j=1}}^{n} r_{ij}\cdot M\left(x_i(t),x_j(t)\right)_{t=T}. \tag{8}$$

After performing the transformation [111], equation (8) will have the form:

$$\begin{aligned}M(\varphi)&=P\left(p_j(0),p_i(0)\right)+\\ &+\int_0^T\int_0^T Q\left(t_1,t_2,p_j(t_1),p_i(t_2)\right)dt_1dt_2,\end{aligned} \tag{9}$$

where $p_j, p_i \in R^n$ satisfies the conditions (5).

The equation of the phase trajectory of the system will be in the form of a differential equation:

$$\dot{y}=B_{n\times n}\left(d_{ij}(t)\right)\cdot y, \tag{10}$$

where B is a matrix that depends on $d_{ij}(t)$; $y^{(i-1)n+s}=p_i^s$, and fulfills the conditions (7).

Accordingly, equation (10) will have the form:

$$M(\varphi)=P(y_0)+\int_0^T\int_0^T Q\left(t_1,t_2,y(t_1),y(t_2)\right)dt_1dt_2, \tag{11}$$

where $p(y_0)=(y,\mathrm{E}y), Q=(y,\Phi y)$, accordingly,

$$\begin{aligned}M(\varphi)&=\left(y(0),\mathrm{E}y(0)\right)+\\ &+\int_0^T\int_0^T Q\left(y(t_1),\Phi(t_1,t_2),y(t_2)\right)dt_1dt_2.\end{aligned} \tag{12}$$

Given the above, the problem of linear synthesis (2)-(3) can be formulated as follows:

1. define control vectors $d_{ij}(t)$;
2. determine the phase trajectory $y(t)$ satisfying the conditions (7), (10) and minimizing the function (12).

Conclusions

The article emphasizes the significance of linear control synthesis within control theory, particularly in addressing dynamic system challenges across various modern technologies and engineering domains. Linear synthesis serves as a pivotal tool in developing optimal control strategies for systems with linear characteristics, offering solutions to complexities in implementing control functions and adapting to changing conditions. It presents equations defining optimal control within a framework of linear functions of phase coordinates, along with conditions and transformations necessary for solving the problem of linear synthesis. Although the problem of linear synthesis has specific aspects of optimal control theory, it is generally quite complex. The complexity of the linear synthesis problem is due to several factors: first, it is significantly nonlinear, requiring special approaches in the theory of optimal control; second, the large size of the parameter space complicates the calculations.

A particular difficulty is that the functionalities we are dealing with do not fall into the usual types studied in calculus of variations and optimal control theory, which complicates the search for a solution. Taking into account all the components of the phase vector, the dimension of the functional state vector increases quadratically with respect to their number. In addition, after discretization, the problem becomes non-additive, and effective solution refinement methods that use the properties of additivity cannot be directly applied in this theory. This requires further research to develop efficient solution methods.

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